SFB-Kolloquium zwischen Potsdam und Berlin

Effective behavior of random media

Max-Planck-Institut für **Mathematik in den**

Naturwissenschaften, Leipzig

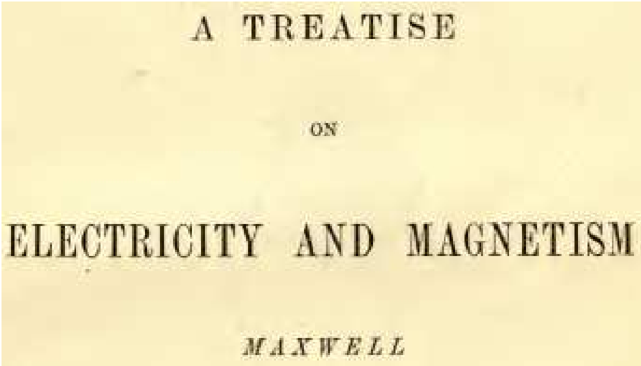
Effective behavior of random media

=Stochastic homogenization:

Early explicit asymptotic treatment,

recent numerical applications

Maxwell: Effective resistance of a composite

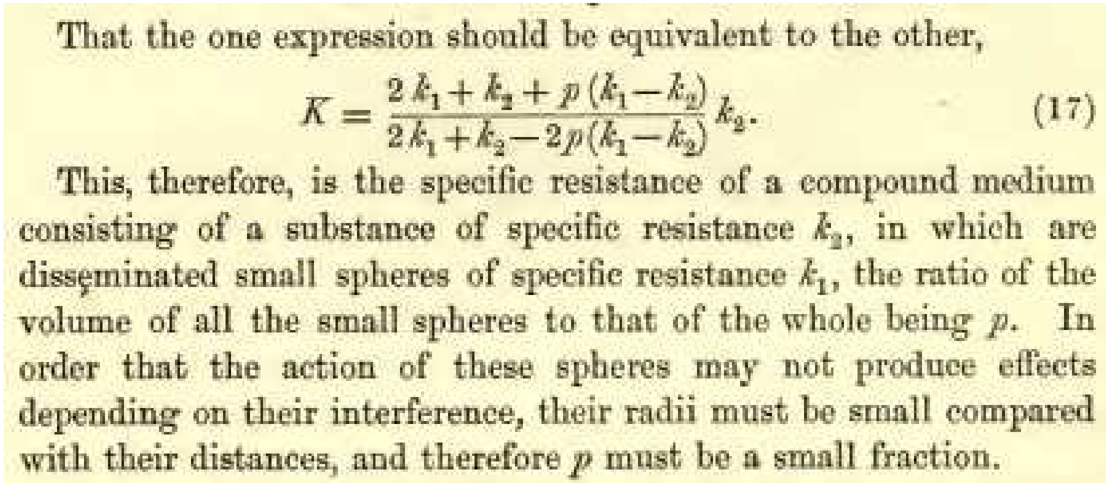


A TREATISE

ON

ELECTRICITY AND MAGNETISM

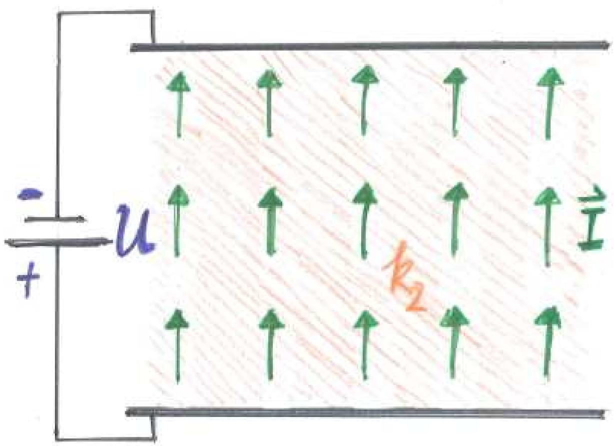
MAXWELL

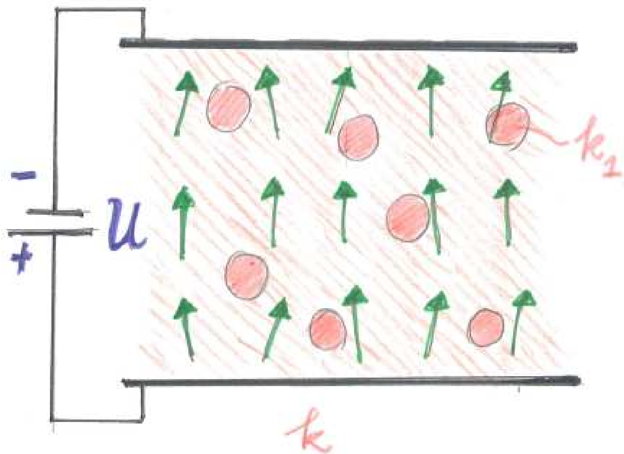


That the one expression should be equivalent to the other,

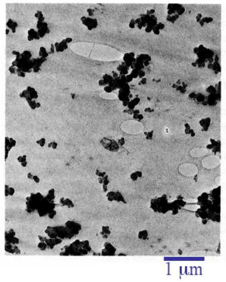
$$K = \frac{2k_1 + k_2 + p(k_1 - k_2)}{2k_1 + k_2 - 2p(k_1 - k_2)} k_2. \quad (17)$$

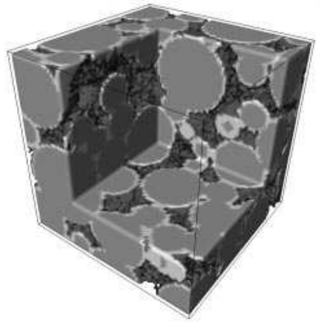
This, therefore, is the specific resistance of a compound medium consisting of a substance of specific resistance k_2, in which are disseminated small spheres of specific resistance k_1, the ratio of the volume of all the small spheres to that of the whole being p. In order that the action of these spheres may not produce effects depending on their interference, their radii must be small compared with their distances, and therefore p must be a small fraction.





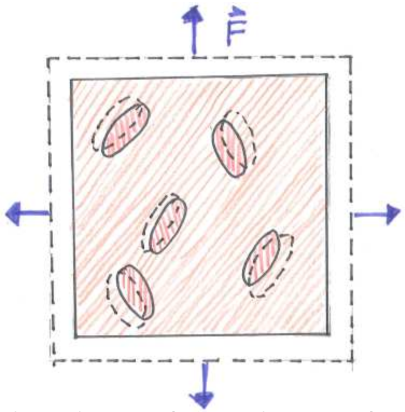
Recent: composite materials & porous media

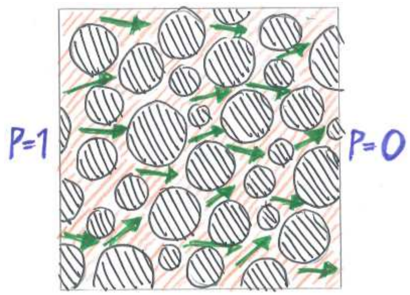




Effective elasticity

Effective permeability





Effective behavior by simulation
of "Representative Volume Element"
Mathematical theory on qualitative level:
Varadhan&Papanicolaou, Kozlov '79, H-convergence by Murat&Tartar

Random medium ...

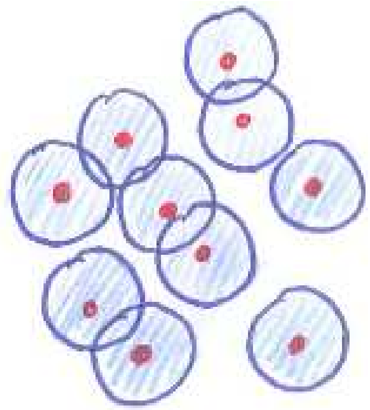
symmetric coefficient field $a = a(x)$ on d-dimensional space

$$\lambda|\xi|^2 \le \xi \cdot a(x)\xi \le |\xi|^2 \quad \text{for all points } x \text{ and vectors } \xi$$

$\rightsquigarrow$ uniformly elliptic operator $-\nabla \cdot a\nabla u$

Ensemble $\langle\cdot\rangle$ of such coefficient fields a

Example of ensemble $\langle\cdot\rangle$:
points Poisson distributed with density 1,
union of balls of radius $\frac{1}{4}$ around points,
$a =$ id on union, $a = \lambda$id on complement,



Stationarity: a and $a(y + \cdot)$ have same distribution under $\langle\cdot\rangle$

... = elliptic operator with random stationary coefficient field

Plan for talk

1) Error in Representative Volume Element (RVE) Method:
Scaling of random and systematic contribution
in terms of RVE-size

2) Fluctuations of macroscopic observables:
leading-order pathwise characterization,
RVE method for extraction

Representative Volume Element method

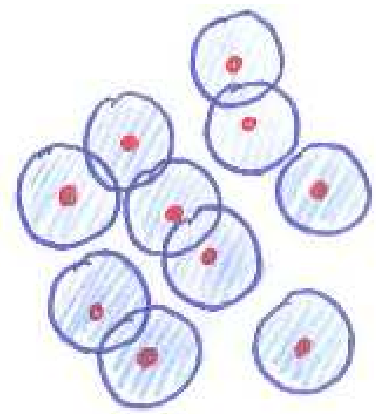
to extract effective tensor $\bar{a}$:

Scaling of random and systematic error in RVE size

A. Gloria, S. Neukamm

Goal: Extract effective behavior $\bar{a}$ from $\langle\cdot\rangle$ ...

Recall example of ensemble $\langle\cdot\rangle$:
points Poisson distributed with density 1,
union of balls of radius $\frac{1}{4}$ around points,
$a = \mathrm{id}$ on union, $a = \lambda\,\mathrm{id}$ on complement,



ensemble $\langle\cdot\rangle$ $\rightsquigarrow$ effective conductivity $\bar{a}$

$$\left\{ \begin{array}{c} \text{density of points } 1 \\ \text{radius of inclusions } \frac{1}{4} \\ \text{conductivity in pores } \lambda \end{array} \right\} \rightsquigarrow \bar{a} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix} = \bar{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3 numbers $\rightsquigarrow$ 1 number

... via Representative Volume Element (RVE)

Representative Volume Element method

Introduce artificial period L

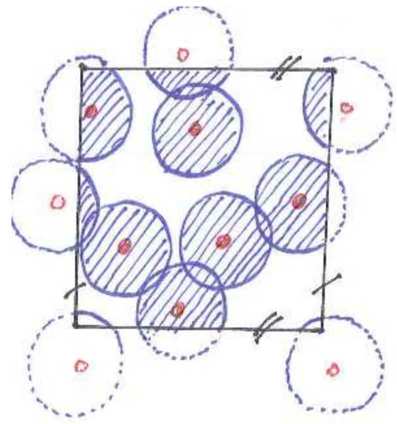
Periodized ensemble $\langle\cdot\rangle_L$

points Poisson distributed with density 1,

on d-dimensional torus $[0,L)^d$

union of balls of radius $\frac{1}{4}$ around points,

$a = \mathrm{id}$ on union, $a = \lambda\,\mathrm{id}$ on complement,



Given coordinate direction $i = 1, \cdots, d$ seek L-periodic φ_i with

$$-\nabla \cdot a(e_i + \nabla\varphi_i) = 0 \quad \text{on } [0,L)^d.$$

Spatial average $\fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$ of flux $a(e_i + \nabla\varphi_i)$

as approximation to $\bar{a}e_i$ for $L \gg 1$;

φ_i is approximate "corrector", e_i unit vector in i-th coordinate direction

Solving d elliptic equations $-\nabla \cdot a(e_i + \nabla\varphi_i) = 0$ ...

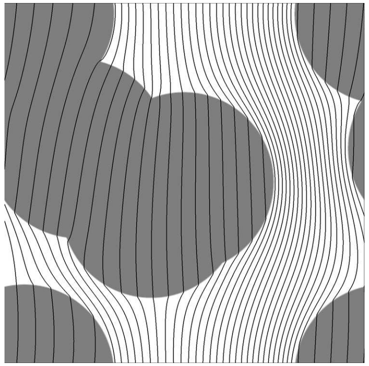
direction e_1

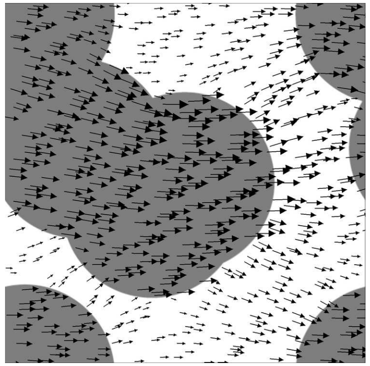
potential

$x_1 + \varphi_1$

flux

$a(e_1 + \nabla\varphi_1)$





average flux

$\fint a(e_1 + \nabla\varphi_1)$

$= \begin{pmatrix} 0.49641 \\ -0.02137 \end{pmatrix}$

$\approx \bar{a} e_1$

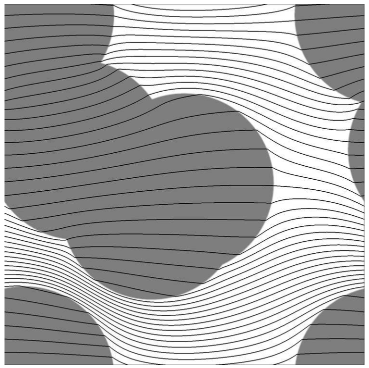
direction e_2

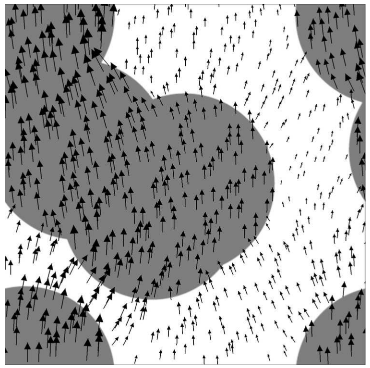
potential

$x_2 + \varphi_2$

flux

$a(e_2 + \nabla\varphi_1)$





average flux

$\fint a(e_2 + \nabla\varphi_2)$

$= \begin{pmatrix} -0.02137 \\ 0.53240 \end{pmatrix}$

$\approx \bar{a} e_2$

simulations by R. Kriemann (MPI)

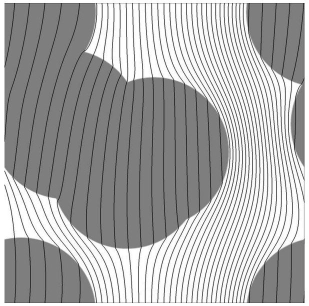
... gives approximation $\bar{a}_L$

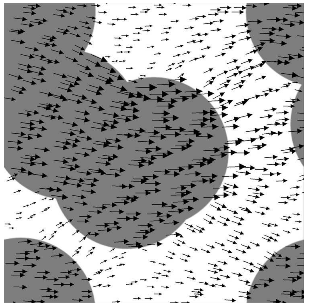
Random error: approx. $\bar{a}_L$ depends on realization

realization 1

potential,

current



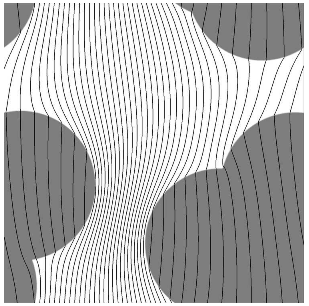


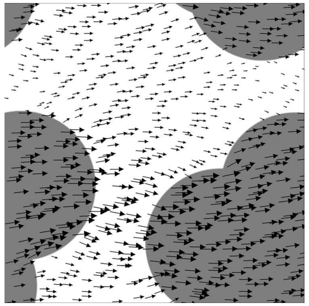
$$\bar{a}_L = \begin{pmatrix} 0.49641 & -0.02137 \\ -0.02137 & 0.53240 \end{pmatrix}$$

realization 2

potential,

current



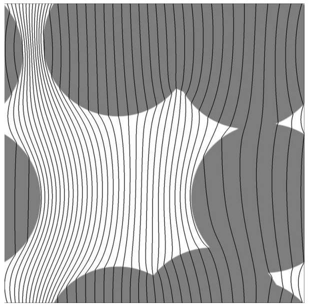


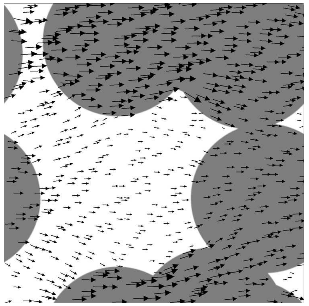
$$\bar{a}_L = \begin{pmatrix} 0.45101 & 0.01104 \\ 0.01104 & 0.45682 \end{pmatrix}$$

realization 3

potential,

current

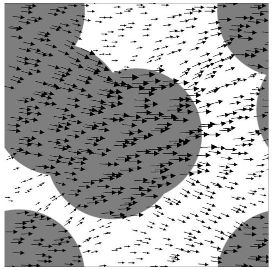




$$\bar{a}_L = \begin{pmatrix} 0.56213 & 0.00857 \\ 0.00857 & 0.60043 \end{pmatrix}$$

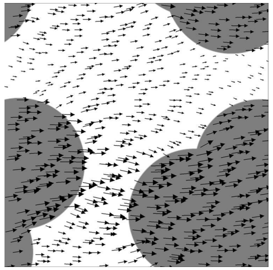
... and thus fluctuates / is random

Fluctuations of $\bar{a}_L$ decrease with increasing L



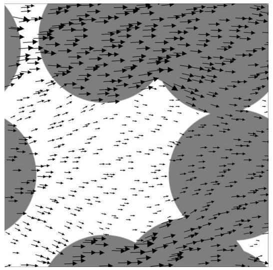
$\bar{a}_L =$

$$\begin{pmatrix} 0.50 & -0.02 \\ -0.02 & 0.53 \end{pmatrix}$$



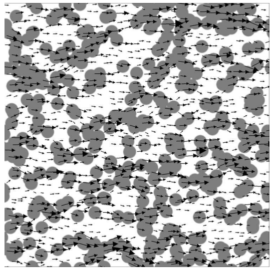
$\bar{a}_L =$

$$\begin{pmatrix} 0.45 & 0.01 \\ 0.01 & 0.46 \end{pmatrix}$$



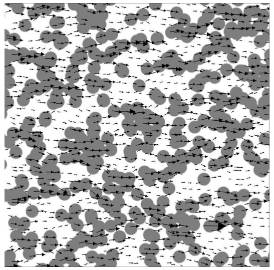
$\bar{a}_L =$

$$\begin{pmatrix} 0.56 & 0.01 \\ 0.01 & 0.60 \end{pmatrix}$$



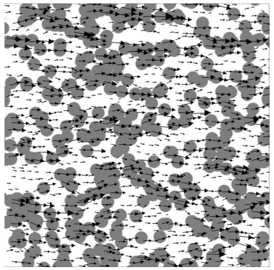
$\bar{a}_L =$

$$\begin{pmatrix} 0.518 & 0.004 \\ 0.004 & 0.511 \end{pmatrix}$$



$\bar{a}_L =$

$$\begin{pmatrix} 0.532 & 0.005 \\ 0.005 & 0.523 \end{pmatrix}$$



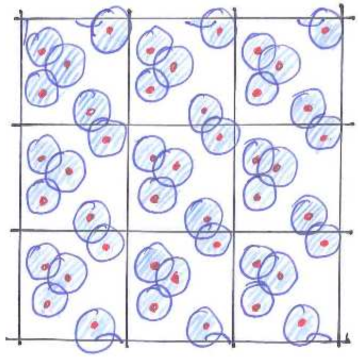
$\bar{a}_L =$

$$\begin{pmatrix} 0.515 & -0.001 \\ -0.001 & 0.521 \end{pmatrix}$$

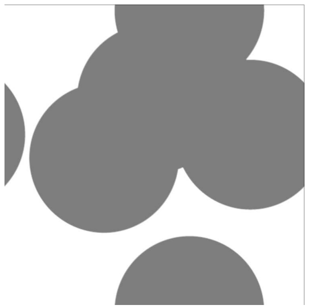
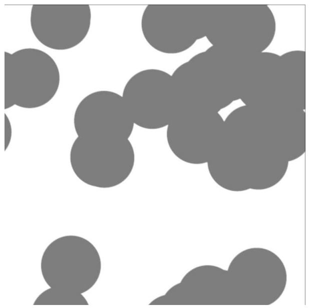
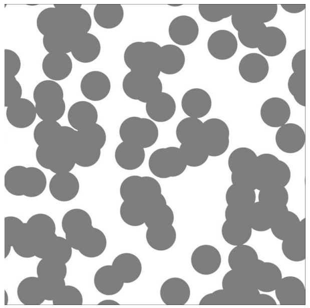
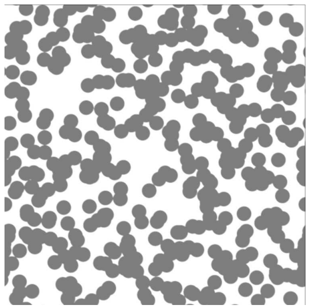
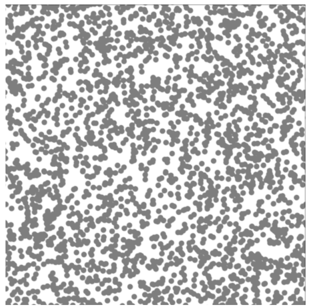
... scaling of variance $\mathrm{var}(\bar{a}_L)$ **in** L**?**

Systematic error, decreases with increasing L

Also expectation $\langle \bar{a}_L \rangle_L$ depends on L since from $\langle \cdot \rangle$ to $\langle \cdot \rangle_L$ statistics are altered by artificial long-range correlations



$$\langle \bar{a}_L \rangle_L = \bar{\lambda}_L \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ because of symmetry of $\langle \cdot \rangle$ under rotation

$L = 2$	$L = 5$	$L = 10$	$L = 20$	$L = 50$
				
0.551	0.524	0.520	0.522	0.522

Scaling of both errors in L ...

Pick a according to $\langle\cdot\rangle_L$, **solve** for φ (period L),

compute spatial average $\bar{a}_L e_i := \fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$

Take *random variable* $\bar{a}_L$ **as approximation to** $\bar{a}$

$\langle$**error**$^2\rangle_L$ = **random**^{2} + **systematic**^{2}:

$$\langle|\bar{a}_L - \bar{a}|^2\rangle_L = \mathrm{var}_{\langle\cdot\rangle_L}[\bar{a}_L] + |\langle\bar{a}_L\rangle_L - \bar{a}|^2$$

Qualitative theory yields:

$$\lim_{L\uparrow\infty} \mathrm{var}_{\langle\cdot\rangle_L}[\bar{a}_L] = 0, \qquad \lim_{L\uparrow\infty} \langle\bar{a}_L\rangle_L = \bar{a}$$

... why rate is of interest?

Number of samples N vs. artificial period L

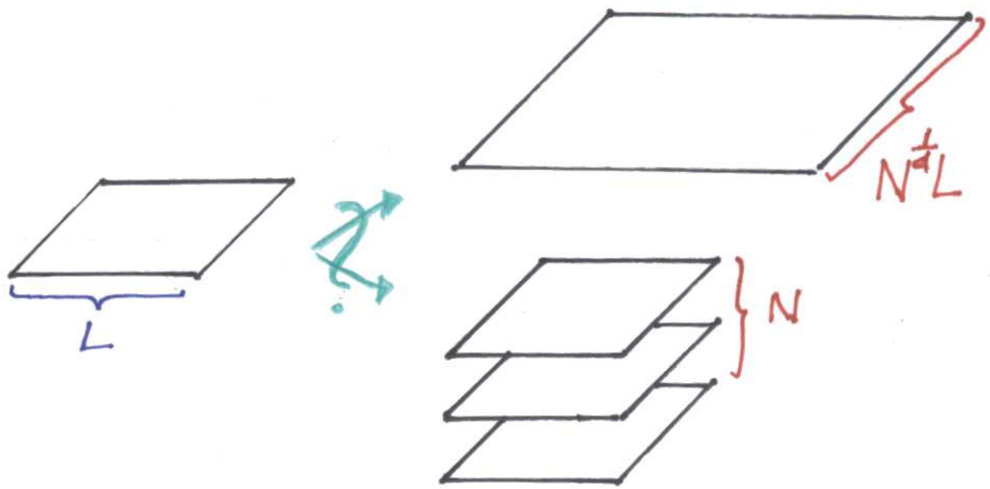
Take **N** samples, i. e. independent picks $a^{(1)},\cdots,a^{(N)}$ from $\langle\cdot\rangle_L$.

Compute empirical mean $\frac{1}{N}\sum_{n=1}^{N}\fint_{[0,L)^d} a^{(n)}(e_i+\nabla\varphi_i^{(n)})$

$$\langle\textbf{total error}^2\rangle_L = \frac{1}{\mathbf{N}}\,\textbf{random error}^2 + \textbf{systematic error}^2$$

L ↑ reduces systematic error and random error

N ↑ reduces only effect of **random error**



An optimal result

Let $\langle\cdot\rangle_L$ be ensemble of a's with period L,
with $\langle\cdot\rangle_L$ suitably coupled to $\langle\cdot\rangle$

For a with period L
solve $\nabla\cdot a(e_i+\nabla\varphi_i)=0$ for φ_i of period L.
Set $\bar{a}_L e_i = \fint_{[0,L)^d} a(e_i+\nabla\varphi_i)$.

Theorem [Gloria&O.'13, G.&Neukamm&O. Inventiones'15]

Random error$^2 = \mathrm{var}_{\langle\cdot\rangle_L}[\bar{a}_L] \;\leq\; C(d,\lambda)L^{-d}$

Systematic error$^2 = \left|\langle\bar{a}_L\rangle_L - \bar{a}\right|^2 \;\leq\; C(d,\lambda)L^{-2d}\ln^d L$

Gloria&Nolen '14: (random) error approximately Gaussian

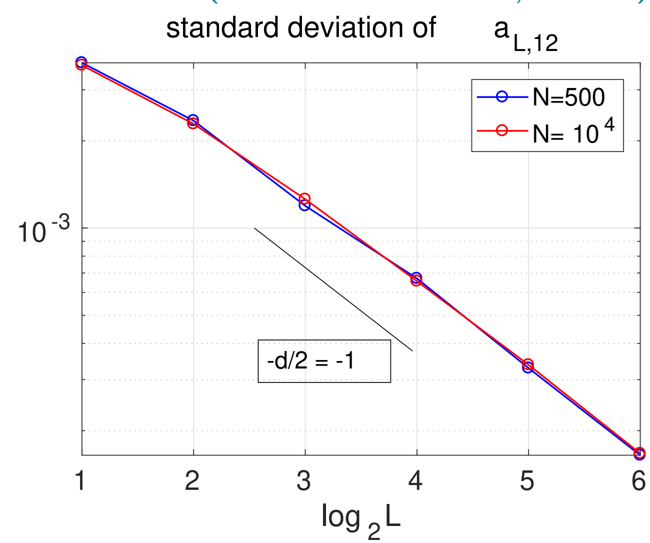
Fischer '17: variance reduction

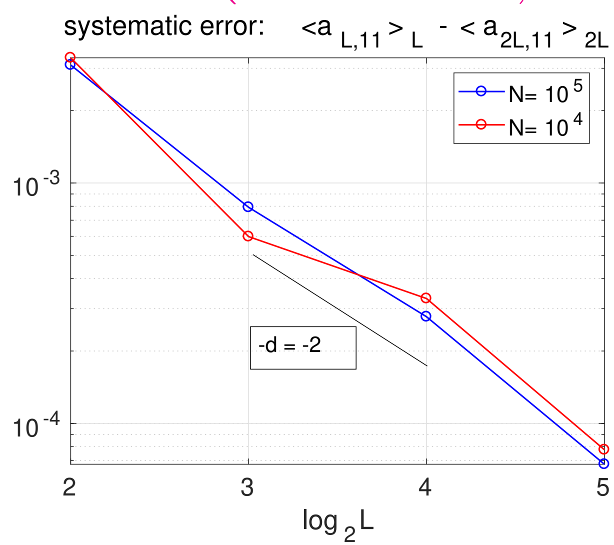
Numerical experiments display optimality

Random error $= \mathrm{var}^{\frac{1}{2}}_{\langle\cdot\rangle_L}[\bar a_L] \ \le\ C(d,\lambda) L^{-\frac{d}{2}}$

Systematic error $= \left|\langle \bar a_L\rangle_L - \bar a\right| \ \le\ C(d,\lambda) L^{-d} \ln^{\frac{d}{2}} L$

$\lim_{N\uparrow\infty}\left(\frac{1}{N}\sum_{n=1}^{N}(\bar a_{L,12}^{(n)})^2\right)^{\frac12}$ $\quad |\lim_{N\uparrow\infty}\left(\frac{1}{N}\sum_{n=1}^{N}\bar a_{L,11}^{(n)} - \frac{1}{N}\sum_{n=1}^{N}\bar a_{2L,11}^{(n)}\right)|$





simulations from Khoromskij&Khoromskaja&Otto for $d = 2$, different ensemble

State of art in quantitative stochastic homogenization ...

Yurinskii '86 : suboptimal rates in L for mixing $\langle\cdot\rangle$

Naddaf & Spencer '98, & Conlon '00:
optimal rates for small contrast $1-\lambda \ll 1$,
for $\langle\cdot\rangle$ with spectral gap

Gloria & O. '11, & Neukamm '13, & Marahrens '13:
optimal rates for all $\lambda > 0$ for $\langle\cdot\rangle$ with spectral gap,
Logarithmic Sobolev (concentration of measure)

Armstrong & Smart '14, & Mourrat '14, & Kuusi '15,
Gloria & O. '15
optimal stochastic integrability for finite range $\langle\cdot\rangle$

... of linear equations in divergence form

Homogenization error on macroscopic observables

Characterization of leading-order variances via a pathwise characterization of leading-order fluctuations

M. Duerinckx, A. Gloria

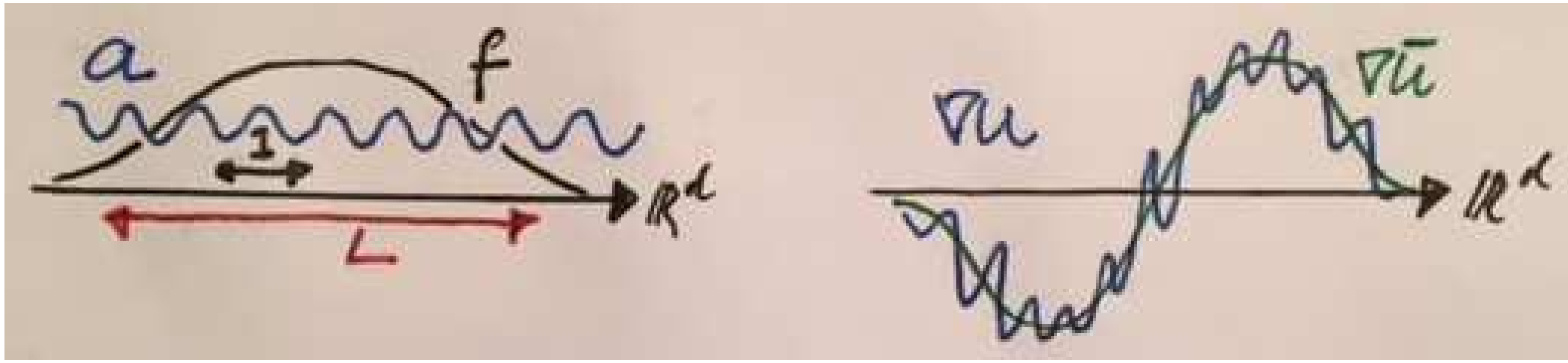
arXiv:1903.02329

Homogenization is based on scale separation

Given $f \in L^2(\mathbb{R}^d)^d$ consider Lax-Milgram solution $\nabla u, \nabla \bar{u}$ solution of
$\nabla \cdot (a\nabla u + f) = 0, \quad \nabla \cdot (\bar{a}\nabla \bar{u} + f) = 0 \quad \text{in } \mathbb{R}^d$

a has microscopic characteristic scale 1,
e. g. correlation length in random case, period in periodic case
f has macroscopic characteristic scale $L \gg 1$,
e. g. $f(x) = \hat{f}(\frac{x}{L})$ for fixed deterministic mask $\hat{f} \in C_0^\infty(\mathbb{R}^d)^d$

Then we want $\nabla u \approx \nabla \bar{u}$ on macroscopic scale L.



Homogenization, general approach

Object of interest: elliptic operator $-\nabla\cdot a\nabla$,
naturally on level of Helmholtz projection $\nabla(\text{-}\nabla\cdot a\nabla)^{-1}\nabla\cdot$,
recall $f\mapsto\nabla u$ where $\nabla\cdot(a\nabla u+f)=0$

Goal: Relate heterogen. a to homogen. $\bar a$ in sense of
$\nabla(\text{-}\nabla\cdot a\nabla)^{-1}\nabla\cdot\approx\nabla(\text{-}\nabla\cdot\bar a\nabla)^{-1}\nabla\cdot$ weakly (macro averages)

Key object: Helmholtz decomposition of $a\text{-}\bar a$:

$$(a\text{-}\bar a)e_i = -a\underbrace{\nabla\phi_i}_{\text{curl-free}}+\underbrace{\nabla\cdot\sigma_i}_{\text{div-free}}$$

for $i=1,\cdots,d$, e_i i-th Cartesian unit vector

in part. $\nabla\cdot a(\nabla\phi_i+e_i)=0$, $x\mapsto x_i+\phi_i(x)$ a-harmonic coordinates

Crucial property: Sublinearity of corrector potentials
ϕ_i (scalar), σ_i (skew-symmetric tensor, i. e. $\sigma_{ijk}=-\sigma_{ikj}$)

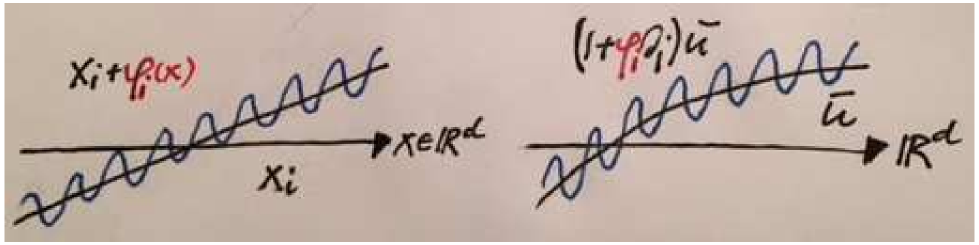
Merit of correctors, two scale expansion

Recall: $(a\text{-}\bar{a})e_i = -a\underbrace{\nabla\phi_i}_{\text{curl-free}} + \underbrace{\nabla\cdot\sigma_i}_{\text{div-free}}$ for $i = 1, \cdots, d$

ϕ_i corrects affine x_i to become a-harmonic;

modulate arbitrary $\bar{u}$ in same way

$\rightsquigarrow$ "two-scale expansion"



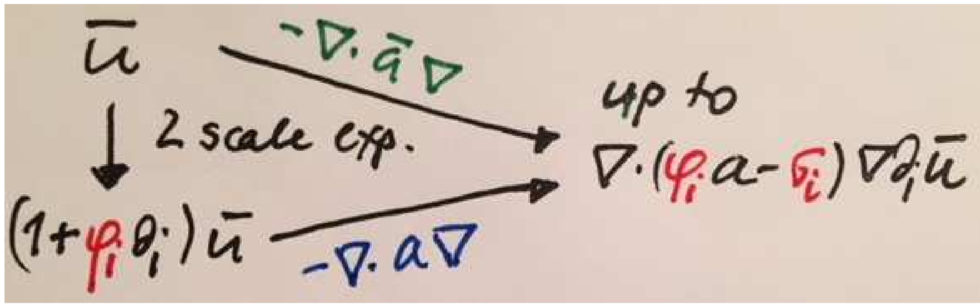
Merit: Relate differential operators $-\nabla\cdot a\nabla$, $-\nabla\cdot\bar{a}\nabla$:

$$-\nabla\cdot a\nabla\underbrace{(1+\phi_i\partial_i)\bar{u}}_{\text{two-scale expansion}} = -\nabla\cdot\bar{a}\nabla\bar{u} + \underbrace{\nabla\cdot(\phi_i a\text{-}\sigma_i)\nabla\partial_i\bar{u}}_{\text{divergence-form}}$$

two-scale expansion: Einstein's summation convention

divergence-form: good for estimate

An almost commuting diagram



Micro oscillations vs. macro fluctuations

For deterministic f consider

$\nabla\cdot(a\nabla u+f)=0$ and $\nabla\cdot(\bar a\nabla\bar u+f)=0$;

where we think of $f(x)=\hat f(\frac{x}{L})$ for $\hat f\in C_0^\infty(\mathbb{R}^d)^d$ deterministic.

Microscopic oscillations:

$\nabla u\approx\nabla(1+\phi_i\partial_i)\bar u$ in strong topology,

i.e. in $(\int|\nabla(u-(1+\phi_i\partial_i)\bar u)|^2)^{\frac{1}{2}}$

Macroscopic fluctuations: $\nabla u\approx\nabla\bar u$ in weak topology,

i. e. in $\int g\cdot(\nabla u-\nabla\bar u)$,

where we think of $g(x)=\frac{1}{L^d}\hat g(\frac{x}{L})$ for $\hat g\in C_0^\infty(\mathbb{R}^d)^d$ deterministic.

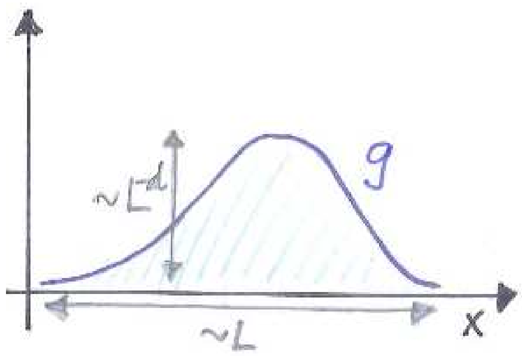
Here: Macroscopic fluctuations

solution ∇u of $\nabla\cdot(a\nabla u+f)=0$,

where r. h. s. $f(x)=\hat{f}(\frac{x}{L})$ deterministic

macroscopic observable $\int g\cdot\nabla u$,

where $g(x)=L^{-d}\hat{g}(\frac{x}{L})$ deterministic



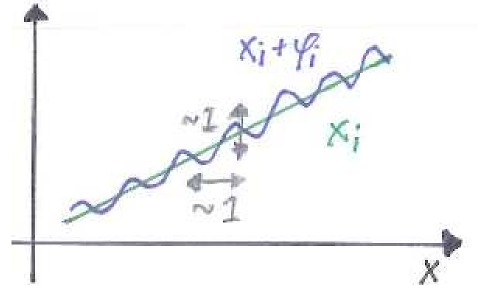
Marahrens & O.'13: $\mathrm{var}(\int g\cdot\nabla u)=O(\frac{1}{L^d})$

Goal: Characterize limiting variance $\lim_{L\uparrow\infty}L^d\mathrm{var}(\int g\cdot\nabla u)$

Naive approach via two-scale expansion

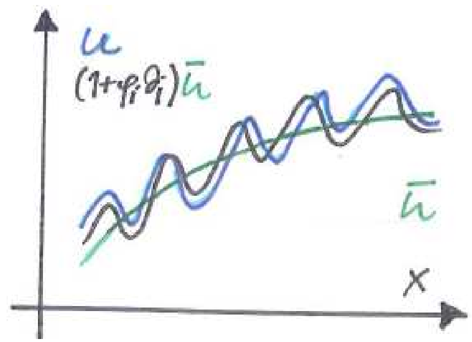
Goal: Characterize limiting variance $\lim_{L\uparrow\infty} L^d \mathrm{var}(\int g\cdot\nabla u)$

Corrector φ_i corrects affine x_i
such that $-\nabla\cdot a(e_i+\nabla\varphi_i)=0$
for coordinate direction $i=1,\cdots,d$



Solution $\bar u$ of homogenized equation
$\nabla\cdot(\bar a\nabla\bar u+f)=0$

Compare u to two-scale expansion
$(1+\varphi_i\partial_i)\bar u$ Einstein's summation rule



Naively expect $\mathrm{var}(\int g\cdot\nabla u) = \mathrm{var}(\int \nabla\cdot g\, u) \approx \mathrm{var}(\int \nabla\cdot g\,(1+\varphi_i\partial_i)\bar u)$
Hence study asymptotic covariance $\langle\varphi_i(x-y)\varphi_j(0)\rangle$

The subtle role of the two-scale expansion

Mourrat&O.'14: $\lim_{L\uparrow\infty} L^{d-2}\langle\varphi_i(L(\hat{x}-\hat{y}))\varphi_j(0)\rangle$ exists,

but $\neq$ a Green function $\bar{G}(\hat{x}-\hat{y})$ (Gaussian free field)

Helffer-Sjöstrand, annealed Green's function bounds $\rightsquigarrow$ 4-tensor $\bar{Q}$

Gu&Mourrat'15: $\lim_{L\uparrow\infty} L^{d}\mathrm{var}(\int g\cdot\nabla u)$ exists,

but $\neq$ $\lim_{L\uparrow\infty} L^{d}\mathrm{var}(\int \nabla\cdot g\,(1+\varphi_i\partial_i)\bar{u})$

Helffer-Sjöstrand $\rightsquigarrow$ same 4-tensor $\bar{Q}$, Gaussianity, heuristics

i. e. two-scale expansion cannot be applied naively

Duerinckx&Gloria&O.'16: Two-scale expansion

$\nabla u\approx\partial_i\bar{u}(e_i+\nabla\varphi_i)$ ok on level of "commutator":

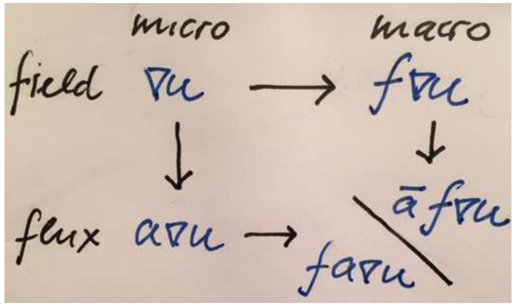
$(a\text{-}\bar{a})\,\nabla u \;\approx\; \partial_i\bar{u}\,(a\text{-}\bar{a})\,(e_i+\nabla\varphi_i)$.

Homogenization commutator is natural

$$(a\text{-}\bar{a})\nabla u = \underbrace{a\nabla u} - \bar{a}\underbrace{\nabla u}$$

flux vs. field

micro vs. macro (=average)



For arbitrary a-harmonic u:

$$e_j \cdot (a\nabla u - \bar{a}\nabla u) = -\nabla \cdot (\phi_j^* a^* - \sigma_j^*)\nabla u,$$

where (ϕ^*, σ^*) corrector of transpose a^*

cf. for arbitrary $\bar{u}$:

$$-\nabla \cdot a\nabla(1+\phi_i\partial_i)\bar{u} = -\nabla \cdot \bar{a}\nabla\bar{u} + \nabla \cdot (\phi_i a - \sigma_i)\nabla\partial_i\bar{u};$$

– a similar algebra for oscillations and fluctuations

Standard homogenization commutator for $u =$ harmonic coord.

$$\Xi e_i := a(\nabla\phi_i + e_i) - \bar{a}(\nabla\phi_i + e_i)$$ stationary tensor field

Leading-order fluctuation of macro observables ...

$\Xi e_i = a(e_i+\nabla\varphi_i) - \bar{a}(e_i+\nabla\varphi_i)$

I) fluctuations commutator $\rightsquigarrow$ fluctuations observable
$\int g\cdot\nabla u = \int \nabla\bar{v}\cdot(a\nabla u - \bar{a}\nabla u) +$ deterministic,
where $\bar{v}$ solves dual equation $\nabla\cdot(\bar{a}^*\nabla\bar{v} + g) = 0$.

II) $a\nabla u$-$\bar{a}\nabla u \approx \Xi\nabla\bar{u}$ holds in quantitative sense of
$L^d\text{var}\big(\int g\cdot(a\nabla u\text{-}\bar{a}\nabla u - \Xi\nabla\bar{u})\big) = O(L^{-2})$.

III) Ξ $\approx$ tensorial white noise holds in quantitative sense of
$L^d\big|\text{var}(\int g\cdot\Xi f) - \int f\otimes g : \bar{Q} f\otimes g\big| = O(L^{-2})$
for four-tensor $\bar{Q}$ from Mourrat&O.

I-III) $L^d\big|\text{var}(\int g\cdot\nabla u) - \int\nabla\bar{v}\otimes\nabla\bar{u} : \bar{Q}\nabla\bar{v}\otimes\nabla\bar{u}\big| = O(L^{-2})$

... characterized via homogenization commutator

How to extract $\bar{Q}$ from $\langle\cdot\rangle$?

Recall standard commutator $\Xi e_i = a(e_i + \nabla\varphi_i) - \bar{a}(e_i + \nabla\varphi_i)$

$$L^d \mathrm{var}\Big(\int g\cdot\nabla u - \int \nabla\bar{v}\cdot\Xi\nabla\bar{u}\Big) = O(L^{-2}), \qquad \nabla\cdot(\bar{a}^*\nabla\bar{v} + g) = 0$$

$$L^d\Big|\mathrm{var}\Big(\int g\cdot\Xi f\Big) - \int f\otimes g : \bar{Q} f\otimes g\Big| = O(L^{-2})$$

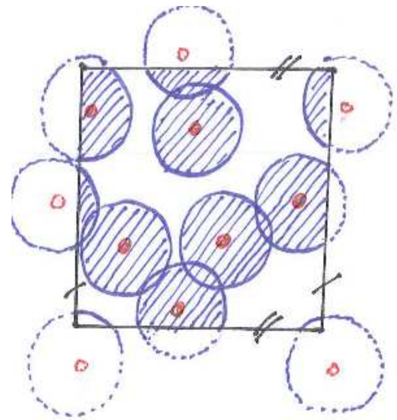
Duerinckx&Gloria&O.'17:

$$|L^d \mathrm{var}_{\langle\cdot\rangle_L}(\bar{a}_L) - \bar{Q}|^2 \le C(d,\lambda) L^{-d} \ln^d L\ ,$$

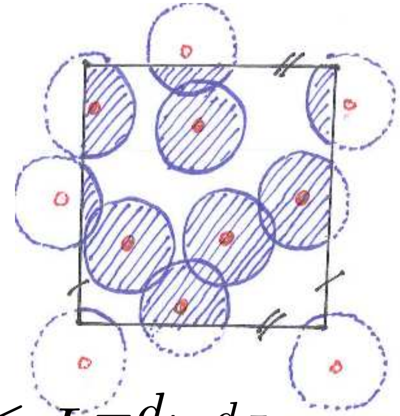
recall: $\langle\cdot\rangle_L$ ensemble of a's with period L,

solve $\nabla\cdot a(e_i + \nabla\varphi_i) = 0$ for φ_i of period L,

Set $\bar{a}_L e_i = \fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$.



In practise: Extract $\bar{Q}$ from RVE ...



Recall periodized ensemble $\langle\cdot\rangle_L$

$$\bar{a}_L e_i = \fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$$

Previous result: $|\langle\bar{a}_L\rangle_L - \bar{a}|^2 \lesssim L^{-2d}\ln^d L$

Duerinckx&Gloria&O.'17: $|L^d \mathrm{var}_{\langle\cdot\rangle_L}(\bar{a}_L) - \bar{Q}|^2 \lesssim L^{-d}\ln^d L$

Hence get $\bar{a}$ and $\bar{Q}$ by same procedure:

$N \sim L^{\frac{d}{2}}$ independent samples $\{a^{(n)}\}_{n=1,\cdots,N}$ from $\langle\cdot\rangle_L$

$$\Big\langle\Big|\frac{1}{N}\sum_{n=1}^{N}\bar{a}_L^{(n)} - \bar{a}\Big|^2\Big\rangle_L \lesssim L^{-2d}\ln^d L,$$

$$\Big\langle\Big|\frac{L^d}{N-1}\sum_{m=1}^{N}\Big(\bar{a}_L^{(m)} - \frac{1}{N}\sum_{n=1}^{N}\bar{a}_L^{(n)}\Big)^{\otimes 2} - \bar{Q}\Big|^2\Big\rangle_L \lesssim L^{-d}\ln^d L$$

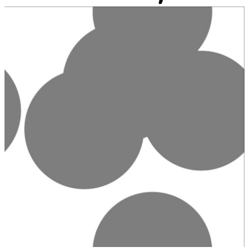
... at no further cost than $\bar{a}$

Back to numerical example

$N \sim L^{\frac{d}{2}}$ independent samples $\{a^{(n)}\}_{n=1,\cdots,N}$ from $\langle\cdot\rangle_L$,

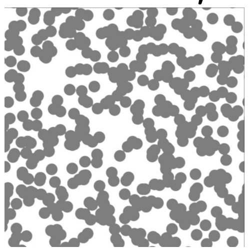
$$\left\langle\left|\frac{L^d}{N-1}\sum_{m=1}^{N}(\bar{a}_L^{(m)}-\frac{1}{N}\sum_{n=1}^{N}\bar{a}_L^{(n)})^{\otimes 2}-\bar{Q}\right|^2\right\rangle_L \lesssim L^{-d}\ln^d L$$

L=2, N=500



$$\bar{Q} = 10^{-2} \times \begin{pmatrix} 1.01 & 0.00 & 0.00 & 0.09 \\ 0.00 & 0.31 & 0.09 & 0.00 \\ 0.00 & 0.09 & 0.31 & 0.00 \\ 0.09 & 0.00 & 0.00 & 0.99 \end{pmatrix}$$

L=20, N=500



$$\bar{Q} = 10^{-2} \times \begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.23 \\ 0.00 & 0.56 & 0.23 & 0.00 \\ 0.00 & 0.23 & 0.56 & 0.00 \\ 0.23 & 0.00 & 0.00 & 1.01 \end{pmatrix}$$

Credits

Gaussianity of various errors: Nolen'14 based on Stein/Chatterjee, Biskup&Salvi&Wolf'14, Rossignol'14, ...

Quartic tensor Q via Helffer-Sjöstrand and Mahrarens& O.'13: Mourrat&O'14, Gu&Mourrat'15

Heuristics of a path-wise approach w/o Ξ: Gu&Mourrat'15, based on variational approach by Armstrong&Smart '13

$\nabla\varphi = \bar{a}$-Helmholtz-projection of white noise: Armstrong&Mourrat&Kuusi'16, Gloria&O.'16
based on finite range rather than Spectral Gap

Commutator & two-scale expansion

Recall: $\langle\cdot\rangle$ ensemble of stationary, centered Gaussian field g on $\mathbb{R}^d$; covariance such that $\mathcal{F}c(k) \le (1+|k|)^{-d-2\alpha}$.

Set $a(x) = A(g(x))$ for smooth A, range in λ-elliptic coefficients.

Recall: $\nabla\cdot(a\nabla u + f) = 0$, $\nabla\cdot(\bar{a}\nabla\bar{u} + f) = 0$,

Standard hom. commutator $\Xi e_i = a(\nabla\phi_i + e_i) - \bar{a}(\nabla\phi_i + e_i)$.

Proposition (Duerinckx&Gloria&O. '14).

$F := \int h\cdot\big(a\nabla u\text{-}\bar{a}\nabla u - \Xi\nabla\bar{u}\big)$ satisfies

$$\langle(F-\langle F\rangle)^{2p}\rangle^{\frac{1}{p}} \lesssim \Big(\int|h|^4\int|\nabla f|^4 + \int|\nabla h|^4\int|f|^4\Big)^{\frac{1}{2}} \quad (d>2)$$

Structure of proof for fluctuations ...

Recall claim: $F=\int h\cdot\big(a\nabla u\text{-}\bar a\nabla u-\Xi\nabla\bar u\big)$ satisfies

$$\langle(F-\langle F\rangle)^{2p}\rangle^{\frac{1}{p}}\lesssim\Big(\int|h|^4\int|\nabla f|^4+\int|\nabla h|^4\int|f|^4\Big)^{\frac{1}{2}}$$

1) L^p-version of spectral gap

$$\langle(F-\langle F\rangle)^{2p}\rangle^{\frac{1}{p}}\lesssim\Big\langle\Big(\int|\frac{\partial F}{\partial a}|^2\Big)^p\Big\rangle^{\frac{1}{p}}$$

2) Representation of Malliavin derivative $\frac{\partial F}{\partial a}$ in terms of solutions to dual problems

3) Annealed Calderon-Zygmund estimate to estimate solutions of dual problems

... as for oscillations

Representation of Malliavin derivative ...

Recall $\nabla\cdot(a\nabla u+f)=0$, $\nabla\cdot(\bar a\nabla\bar u+f)=0$, f,h deterministic.

Recall $F=\int h\cdot(a-\bar a)\big(\nabla u-\partial_i\bar u(e_i+\nabla\phi_i)\big)$

Recall $\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}(F(a+\epsilon\delta a)-F(a))=\int\frac{\partial F}{\partial a}(a,x)\delta a(x)dx$

$$\begin{aligned}\frac{\partial F}{\partial a} &= h_j(e_j+\nabla\phi_j^*)\otimes(\nabla w+\phi_i\nabla\partial_i\bar u)\\ &+ (\nabla w^*+\phi_j^*\nabla h_j)\otimes\nabla u\\ &-(\nabla w_i^*+\phi_j^*\nabla(h_j\partial_i\bar u))\otimes(e_i+\nabla\phi_i),\end{aligned}$$

where
$$\begin{aligned}\nabla\cdot(a\,\nabla w+(\phi_i\,a\,\text{-}\sigma_i)\nabla\,\partial_i\bar u\,)&=0,\\ \nabla\cdot(a^*\nabla w^*+(\phi_j^*a^*\text{-}\sigma_j^*)\nabla h_j\,)&=0,\\ \nabla\cdot(a^*\nabla w_i^*+(\phi_j^*a^*\text{-}\sigma_j^*)\nabla(h_j\partial_i\bar u))&=0.\end{aligned}$$

... via solutions to dual equations

Annealed Calderon-Zygmund estimate

Recall auxiliary problems from representation:

$$\nabla\cdot(a\ \nabla w\ +(\phi_i a\ -\sigma_i)\nabla\quad \partial_i\bar u\ )=0,$$
$$\nabla\cdot(a^*\nabla w^*+(\phi_j^* a^*-\sigma_j^*)\nabla h_j\qquad)=0,$$
$$\nabla\cdot(a^*\nabla w_i^*+(\phi_j^* a^*-\sigma_j^*)\nabla(h_j\partial_i\bar u))=0.$$

Lemma (Duerinckx&O.'19)

Suppose that $\nabla\cdot(a\nabla w+h)=0$; then for all $p,q,q'\in(1,\infty)$

$$\Big(\int\langle|\nabla w|^q\rangle^{\frac{p}{q}}\Big)^{\frac{1}{p}}\lesssim\Big(\int\langle|h|^{q'}\rangle^{\frac{p}{q'}}\Big)^{\frac{1}{p}}\quad\text{provided } q<q'.$$

Maximal regularity in $L^p(\mathbb{R}^d,L^q(\langle\cdot\rangle))$,

no loss in spatial p, tiny (unavoidable) loss in stochastic q.

Relies on stochastic estimates of (ϕ,σ)
and on quenched large-scale Calderon-Zygmund

Summary

Quantitative stochastic homogenization

Goals:

numerical analysis of Representative Volume Element method,
estimate of homogenization error:
[oscillations (micro/strong)], fluctuations (macro/weak)

Concepts: two-scale expansion, flux correctors
homogenization commutator

Tools: spectral gap estimate,
representation formula for Malliavin derivative,
annealed Calderon-Zygmund estimates