Introduction to the Theory of Gibbs Point Process

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lecture 1

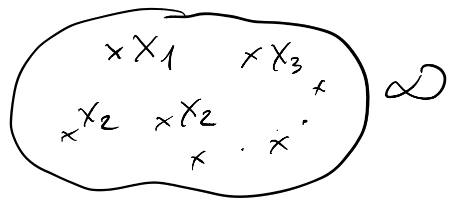
I) Random Point configurations

$\mathcal{D} \subset \mathbb{R}^d$ bounded

$X_1, X_2, \dots, X_n$ iid r.v.

$X_i \in \mathcal{D}$

indistinguishable points



$\Omega = \{ \gamma \subset \mathcal{D}, \ \# \gamma < +\infty \}, \mathcal{F}_\Omega, \mathbb{P}$

Γ a random element of Ω is a point process.

$\mathbb{P}_\Gamma$ the distribution of Γ

Marks: $\gamma \subset \mathcal{D} \times M$ $\times$

Reference Model: Poisson Point Process

Gibbs Point Process is the largest class of model

Lecture 1: P.P.P

Lecture 2: finite Volume Gibbs P.P.

Interaction between the points

Lecture 3: Estimation of parameters

MLE, PMLE, ...

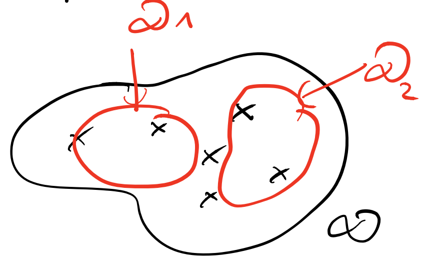
asymptotic: $\mathcal{D} \to \mathbb{R}^d$

Lecture 4: Infinite volume Gibbs P.P

$\mathcal{D} \to \mathbb{R}^d$

Phase Transition: Liquid - gas / Liquid - solid

II) Poisson Point Process



$\mathcal{D} \subset \mathbb{R}^d$ bounded

- uniform scattered in $\mathcal{D}$
- most independent as possible

z = mean number of points per unit volume.

$N = [z \,\text{Volume}(\mathcal{D})]$

$X_1, \dots, X_N$ iid variables uniformly distributed in $\mathcal{D}$

$$\Gamma = \{X_1, X_2, \dots, X_N\}$$

N random, $N \sim \text{Poisson}(\lambda = z \,\text{Volume}(\mathcal{D}))$

$$\left(\text{i.e } \underset{k \geq 0}{P(N=k)} = e^{-\lambda} \frac{\lambda^k}{k!}\right)$$

$(X_i)_{i \geq 0}$ iid $\mathcal{U}(\mathcal{D})$.

def
$$P.P.P_{\mathcal{D}}(z) = \{X_1, X_2, \dots, X_N\}$$

Poisson Point Process

Proposition: Γ a Poisson Point Process with intensity $z > 0$ in a domain $\mathcal{D}$. $\mathcal{D}_1 \subset \mathcal{D}$, $\mathcal{D}_2 \subset \mathcal{D}$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$.

$\longrightarrow$ $N(\Gamma_{\mathcal{D}_1}) \sim \text{Poisson}(z \,\text{Vol}(\mathcal{D}_1))$

$\longrightarrow$ $\Gamma_{\mathcal{D}_1}$ and $\Gamma_{\mathcal{D}_2}$ are independent

P.P.P in $\mathbb{R}^d$

$\mathbb{R}^d$

$$\left.\begin{array}{l}\text{P.P.P in } C_1 \\ \text{P.P.P in } C_2\end{array}\right\} \text{ind}$$

$$\Gamma = \bigcup_{k \in \mathbb{Z}^d} \Gamma_k$$

$k \in \mathbb{Z}^d$

$(\Gamma_k)_{k \in \mathbb{Z}^d}$ ind P.P.P with Γ_k P.P.P in C_k

III) Mecke - Slyvniak Formula

Γ P.P.P in ~~$\mathbb{R}^d$~~ $\mathcal{D}$ with intensity $z > 0$

a function $f : \cancel{\mathbb{R}^d}\, \mathcal{D} \times \Omega_{\mathcal{D}} \longrightarrow \mathbb{R}^+$

Formula:

$$\boxed{\mathbb{E}\Big(\sum_{x \in \Gamma} f(x, \Gamma \setminus \{x\}) \Big) = z\, \mathbb{E}\Big(\int_{\cancel{\mathbb{R}^d}\, \mathcal{D}} f(x, \Gamma)\, dx \Big)}$$

application: • $f(x, \Gamma) = \mathbb{1}_{\mathcal{D}}(x)$

$$\mathbb{E}\big(N_{\mathcal{D}}(\Gamma) \big) = z \,\text{Volume}(\mathcal{D})$$

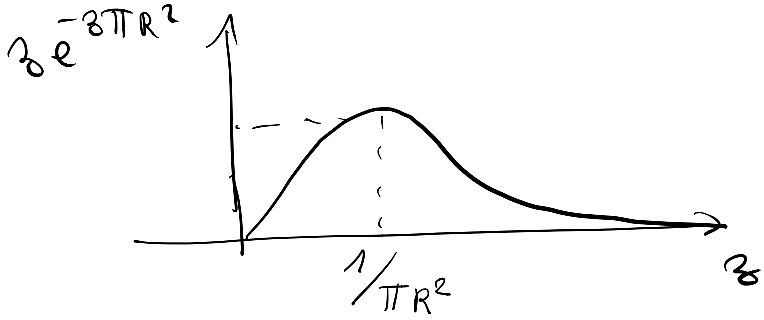
• $R > 0$

$$f(x, \Gamma) = \mathbb{1}_{\mathcal{D}}(x)\, \mathbb{1}_{\{\Gamma \cap B(x, R) = \emptyset\}}$$

$$\mathbb{E}\big(\text{Number of isolated Points in } \Gamma_{\mathcal{D}} \big)$$

$$= z \,\text{Vol}(\mathcal{D})\, P(0 \text{ is isolated in } \Gamma)$$

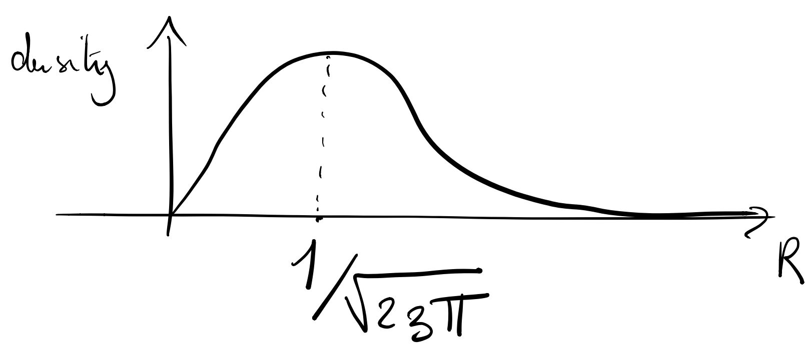
$(d = 2)$ $$= z \,\text{Vol}(\mathcal{D})\, e^{-z \pi R^2}$$



$$z e^{-z\pi R^2} = \mathbb{P}\left(\begin{array}{l} \text{the nearest point of a typical} \\ \text{point is further than } R \end{array} \right)$$

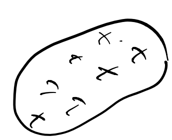
$$R \to 1 - z e^{-z\pi R^2}$$

$$\frac{\partial \left(1 - z e^{-z\pi R^2}\right)}{\partial R} = 2 z^2 \pi R e^{-z\pi R^2}$$



$\boxed{d=2}$

$\mathbb{R}^d$



Finite Volume Gibbs Point Process

Lecture 2

· Lectures on Poisson P. P. 2017 Last - Penrose

I) Energy function

$$\Omega_f = \{ \gamma \subset \mathbb{R}^d \quad \# \gamma < +\infty \}$$

Def: $$H : \Omega_f \longrightarrow \mathbb{R} \cup \{+\infty\}$$
$$\gamma \longmapsto H(\gamma)$$

Assumptions:

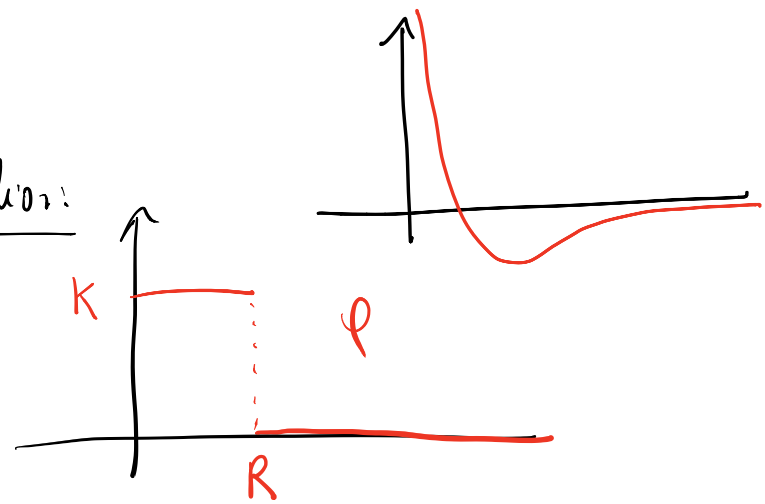
- $H(\emptyset) = 0$
- H is stable, $\exists A \geq 0$, $\forall \gamma \quad H(\gamma) \geq -A \#(\gamma)$
- H is hereditary: $\forall x \in \mathbb{R}^d$, $\forall \gamma \in \Omega_f$
$$H(\gamma) = +\infty \implies H(\gamma \cup x) = +\infty$$

example: • pairwise interaction:

$$H(\gamma) = \sum_{x \neq y \in \gamma} \varphi(|x - y|)$$

Lennard - Jones : $\varphi(r) = a r^{-12} + b r^{-6}$

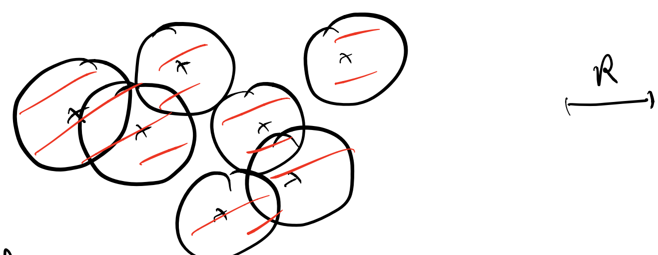
Strauss ~~mx~~ interaction:



- Area - interaction

$R > 0$

$$H(\gamma) = \text{Volume}\Big(\bigcup_{x\in\gamma} B(x,R)\Big)$$



II) Finite Volume GPP

$\Lambda \subset \mathbb{R}^d$ bounded

Definition a GPP in Λ for the activity $z>0$ and inverse of temperature $\beta \geq 0$ is the probability

$$\mathbb{P}_\Lambda^{z,\beta} = \frac{1}{Z_\Lambda^{z,\beta}} e^{-\beta H} \pi_\Lambda^z$$

$\leftarrow$ P.P.P(z) on Λ

$$Z_\Lambda^{z,\beta} = \int e^{-\beta H} \pi_\Lambda^z$$

Remark: $Z_\Lambda^{z,\beta} > 0$ and $Z_\Lambda^{z,\beta} < +\infty$

$\boxed{\begin{array}{l}\textbf{Proposition: } \cdot \text{ The function } z \longmapsto \mathbb{E}_{P_\Lambda^{z,\beta}}(N) \\ \text{is } \mathcal{C}^1 \text{ with derivative } = \frac{1}{z}\mathrm{Var}_{P_\Lambda^{z,\beta}}(N) > 0 \\ \cdot \text{ The function } \beta \longmapsto \mathbb{E}_{P_\Lambda^{z,\beta}}(H) \\ \text{is } \mathcal{C}^1 \text{ with derivative } = -\mathrm{Var}_{P_\Lambda^{z,\beta}}(H) < 0\end{array}}$

III) Variational Principle

2 probability measures μ, ν

$$I(\mu|\nu) = \begin{cases} \int \ln\left(\frac{\mu}{\nu}\right) d\mu & \text{if } \mu \ll \nu \\ +\infty & \text{otherwise} \end{cases}$$

entropy

$\boxed{\begin{array}{l}\textbf{Proposition:} \\ \{P_\Lambda^{z,\beta}\} = \underset{P \in \mathcal{P}_\Lambda}{\mathrm{argmin}} \quad \beta E_P(H) + I(P | \pi_\Lambda^z)\end{array}}$

IV) Simulation

- MCMC algo: birth-Death process
- exact algo (Propp-Wilson)
  Moller - Van lieshout

V) DLR equations

VI) GNZ equations

(Georgii - Nguyen - Zessin)

Definition the local energy function:

$$h : \mathbb{R}^d \times \Omega_f \longrightarrow \mathbb{R} \cup \{+\infty\}$$
$$(x, \gamma) \longmapsto H(\gamma \cup \{x\}) - H(\gamma)$$

$+\infty - +\infty = 0$

Theorem: $f : \mathbb{R} \times \Omega_f \to \mathbb{R}^+$

$$\mathbb{E}_{P_\Lambda^{z,\beta}}\Big(\sum_{x\in\Gamma} f(x, \Gamma\setminus\{x\})\Big) = z\,\mathbb{E}_{P_\Lambda^{z,\beta}}\Big(\int_\Lambda f(x,\Gamma)\underbrace{e^{-\beta h(x,\Gamma)}}_{\neq \text{ Slivnyak-Mecke formula}} dx\Big)$$

Remark: $Z_\Lambda^{z,\beta}$ has disappeared.

$$f(x,\Gamma) = \mathbb{1}_{\Lambda}(x)$$

$$E_{P_\Lambda^{z,\beta}}(N_\Lambda) = z\, E_{P^{z,\beta}}\left(\int_\Lambda e^{-\beta h(x,\gamma)}\, dx\right)$$

$$(\text{if } \quad a < h < b) \qquad \leq z \,\text{Volume}(\Lambda)\, e^{-\beta a}$$

$$\text{or} \quad \geq z\, \text{Volume}(\Lambda)\, e^{-\beta b}$$

$$\boxed{\begin{array}{l}\underline{\text{Propositio}}: \text{ if } \quad a \leq h \leq b \quad \text{then} \\ \pi_\Lambda^{z e^{-\beta b}} \ll P_\Lambda^{z,\beta} \ll \pi_\Lambda^{z e^{-\beta a}}\end{array}}$$

There exists a realization of $\Gamma_1, \Gamma_2, \Gamma_3$ Process.

$$\Gamma_1 \overset{d}{\sim} \pi_\Lambda^{z e^{-\beta b}}$$

$$\Gamma_2 \overset{d}{\sim} P_\Lambda^{z,\beta}$$

$$\Gamma_3 \overset{d}{\sim} \pi_\Lambda^{z e^{-\beta a}}$$

with $\quad \Gamma_1 \subset \Gamma_2 \subset \Gamma_3$